Computing Debt-Limit Contractions in Continuous-Time Heterogeneous-Agent Models

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Abstract

This paper develops a method for computing the transitional dynamics that follow a tightening of the debt limit in heterogeneous-agent models in continuous time (HACT). A contraction shifts the borrowing limit inward, leaving some households with legacy debt outside the newly admissible state space. The method exploits state boundary constraints on the original wealth grid and modifies the upwind scheme to ensure that households with legacy debt adjust toward the new borrowing limit. We apply the method to a benchmark incomplete-markets economy and show how debt limit contractions reshape the wealth distribution and equilibrium prices over time. Although a tighter limit forces households with legacy debt to rebuild net worth more rapidly, requiring them to sacrifice consumption during the adjustment, the induced changes in wages and interest rates can more than offset such costs for moderate contractions and raise aggregate transition welfare.

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1 Introduction

Achdou et al. (2022) show that continuous-time heterogeneous-agent models (HACT) can be formulated as a mean-field game characterized by a coupled Hamilton–Jacobi–Bellman (HJB) and Kolmogorov forward equation (KFE) system. In its standard implementation, their framework allows transitions following unexpected temporary or permanent parameter changes.[1] Although the method accommodates changes in many model parameters, the transition algorithm is formulated on a fixed state space and does not address contractions of the borrowing limit. The borrowing limit determines the lower boundary of the wealth grid. Consequently, the initial and terminal steady states are defined on different grids, which complicates the computation of the transition between them. This paper proposes a fixed-grid solution that exploits two central elements of the numerical method in Achdou et al. (2022): state boundary constraints and the upwind finite-difference scheme.

The method retains the asset grid associated with the least restrictive borrowing limit throughout the transition. As the limit tightens, state boundary constraints are imposed at the grid points affected by the contraction. The upwind scheme is modified to ensure that the finite-difference derivative of the value function is consistent with the direction of the wealth drift implied by these constraints. The initial and terminal steady states and the transitional HJB–KFE system can therefore be computed on the same numerical grid, allowing the initial distribution to be evolved forward while tracking households located between the old and new borrowing limits during the transition.[2]

The application also extends the analysis in Obiols-Homs (2011), who studies the transitional welfare effects of relaxing the borrowing limit and, through steady-state comparisons, shows that tighter limits can raise aggregate welfare in a production economy through their effects on prices. We complement these findings by computing the transition to a tighter borrowing limit and evaluating its welfare consequences once the associated adjustment costs are taken into account. We show that, under the benchmark calibration, general equilibrium price effects can outweigh the costs of adjustment, yielding aggregate welfare gains in moderate contractions. The application thereby provides a continuous-time counterpart to the analysis of household deleveraging in Guerrieri and Lorenzoni (2017), and connects to the literature on borrowing limits and welfare (Obiols-Homs, 2011; Dávila et al., 2012; Nuño and Moll, 2018; Mellior, 2025).

Section 2 sets up the model. Section 3 develops the solution method for debt limit contractions. Section 4 presents the numerical results. Section 5 concludes.

2 Environment

The economy is a standard Aiyagari (1994) incomplete-markets model, written in continuous time following Achdou et al. (2022). There is a unit mass of infinitely lived households who maximize expected utility from consumption, c_t, discounted at rate ρ. Households differ in wealth b and idiosyncratic labor productivity z_i. Productivity follows a finite-state Poisson process, where λ_{ij} denotes the arrival rate of a transition from state i to state j. A household supplies labor inelastically and can save by accumulating wealth. Negative wealth denotes debt. There is an exogenous debt limit, $\underline{b}_t < 0$, such that $b_t \geq \underline{b}_t > -\frac{wz_L}{r_t}$, where wz_L is the lowest possible labor income and r is the interest rate.[3] The value of a household with wealth b and

[1] These are commonly referred to as MIT shocks.

[2] Under an immediate contraction, the initial distribution may assign positive mass to asset positions below the new limit. Under a gradual contraction, the boundary moves through the grid over time. The fixed-grid formulation accommodates both cases.

[3] Note that the debt limit may be time-varying.

productivity state i at time t is

$$V_i(b_t) = \max_{\{c_s\}_{s\geq t}} \mathbb{E}_t \left[\int_t^{\infty} e^{-\rho(s-t)} u(c_s)\, ds \right], \tag{1}$$

subject to

$$\frac{\mathrm{d}b_t}{\mathrm{d}t} = w_t z_{it} + r_t b_t - c_t, \qquad b_t \geq \underline{b}_t, \tag{2}$$

where w_t is the wage and r_t is the interest rate. Preferences are given by

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0. \tag{3}$$

For notational convenience we drop time subscripts and define the wealth drift of a household in productivity state i as

$$S_i(b) = wz_i + rb - c. \tag{4}$$

The HJB equation is

$$\rho V_i(b) = \max_c \left\{ u(c) + \frac{\partial V_i(b)}{\partial b} S_i(b) + \sum_{j\neq i} \lambda_{ij} \left[V_j(b) - V_i(b) \right] + \frac{\partial V_i(b)}{\partial t} \right\}. \tag{5}$$

The first-order condition for consumption is

$$u'(c_i(b)) = \frac{\partial V_i(b)}{\partial b}. \tag{6}$$

At the borrowing limit, the state constraint requires

$$S_i(\underline{b}) \geq 0. \tag{7}$$

Let $g_i(b)$ denote the density of households with wealth b and productivity state i at time t. The density evolves according to the KFE

$$\frac{\partial g_i(b)}{\partial t} = -\frac{\partial}{\partial b} \left[S_i(b) g_i(b) \right] + \sum_{j\neq i} \left[\lambda_{ji} g_j(b) - \lambda_{ij} g_i(b) \right]. \tag{8}$$

A representative firm produces output using capital and effective labor according to

$$Y_t = A K_t^{\alpha} L_t^{1-\alpha}, \tag{9}$$

where A is total factor productivity and α is the capital share. Aggregate effective labor is

$$L_t = \sum_i z_i \int g_i(b)\, db. \tag{10}$$

Competitive factor prices are

$$r_t = \alpha A \left(\frac{K_t}{L_t} \right)^{\alpha-1} - \delta, \qquad w_t = (1-\alpha) A \left(\frac{K_t}{L_t} \right)^{\alpha}, \tag{11}$$

where δ is the depreciation rate. The asset market clears when household wealth equals the aggregate capital stock:

$$K_t = \sum_i \int b\, g_i(b)\, db. \tag{12}$$

3 Implementing borrowing-limit contractions

Achdou et al. (2022) represent the heterogeneous-agent equilibrium as a coupled HJB–KFE system. Policy functions are obtained from an upwind discretization of the HJB equation, using a forward difference when the wealth drift is positive and a backward difference when it is negative. At the borrowing limit, a state boundary condition supplies the derivative that prevents households from drifting outside the state space. The method proposed below retains these ingredients but adapts them to a setting in which the borrowing limit, and hence the location of the state boundary, changes during the transition. Let $D_b^B V_{i,t}(b)$ and $D_b^F V_{i,t}(b)$ denote the backward and forward finite-difference approximations to the wealth derivative of $V_{i,t}(b)$. The associated consumption policies and drifts are denoted by $c_{i,t}^B(b)$, $c_{i,t}^F(b)$, $S_{i,t}^B(b)$, and $S_{i,t}^F(b)$. At the lower boundary $\underline{b}$, the household cannot choose a negative wealth drift:

$$S_{i,t}(\underline{b}) \geq 0.$$

If this condition binds, consumption is restricted to available resources,

$$c_{i,t}^B(\underline{b}) = w_t z_i + r_t \underline{b},$$

and the corresponding backward derivative is assigned through

$$D_b^B V_{i,t}(\underline{b}) = u'(w_t z_i + r_t \underline{b})\,.$$

Now consider a permanent tightening from $\underline{b}_0$ to $\underline{b}_T > \underline{b}_0$. The numerical grid continues to begin at $\underline{b}_0$ throughout the transition. Let $\underline{b}_t \in [\underline{b}_0, \underline{b}_T]$ denote the operative borrowing limit at date t, and define $\mathcal{L}_t = [\underline{b}_0, \underline{b}_t)$ as the region of the fixed grid lying below the time-t borrowing limit. The policy is implemented by activating additional state boundary constraints at the points in $\mathcal{L}_t$. Let $\widetilde{S}_{i,t}(b) \geq 0$ denote the minimum wealth drift required at an affected grid point. The extended state constraint is

$$S_{i,t}(b) \geq \begin{cases} \widetilde{S}_{i,t}(b), & b < \underline{b}_t, \\ 0, & b = \underline{b}_t. \end{cases} \tag{13}$$

The path of $\underline{b}_t$ determines when the additional constraints become active, while $\widetilde{S}_{i,t}(b)$ determines how rapidly households must move toward the admissible region.[4] When the constraint binds below the operative limit, consumption and the backward value derivative satisfy

$$c_{i,t}^B(b) = w_t z_i + r_t b - \widetilde{S}_{i,t}(b), \qquad D_b^B V_{i,t}(b) = u'\Big(w_t z_i + r_t b - \widetilde{S}_{i,t}(b)\Big)\,. \tag{14}$$

Households remain free to save faster whenever their unconstrained policy implies a larger positive drift. The upwind step requires a further modification. For a concave value function, the one-sided approximations typically satisfy

$$D_b^B V_{i,t}(b) > D_b^F V_{i,t}(b) \quad \Rightarrow \quad c_{i,t}^B(b) < c_{i,t}^F(b) \quad \Rightarrow \quad S_{i,t}^B(b) > S_{i,t}^F(b). \tag{15}$$

It is therefore possible to obtain $S_{i,t}^B(b) > 0 > S_{i,t}^F(b)$. In the standard upwind scheme, neither directional derivative is then selected and the grid point is treated as a temporary steady state with zero drift. This treatment is inconsistent with the contraction policy whenever the state constraint requires $S_{i,t}(b) \geq \widetilde{S}_{i,t}(b) > 0$. Although the required drift is introduced through the backward derivative, a positive drift is implemented by the upwind scheme using the forward

[4] Different choices of $\widetilde{S}_{i,t}(b)$ represent different treatments of outstanding debt. Setting $\widetilde{S}_{i,t}(b) = 0$ prevents households from taking on additional debt but allows existing balances to be repaid endogenously. A strictly positive value imposes minimum amortization, and the required rate may vary with wealth, income or calendar time.

derivative. Modifying only the backward candidate is therefore insufficient. Let $\widehat{D_b^F V_{i,t}}(b)$ denote the forward derivative before the correction. At grid points in $\mathcal{L}_t$, we replace it with

$$D_b^F V_{i,t}(b) = \max\left\{\widehat{D_b^F V_{i,t}}(b), D_b^B V_{i,t}(b)\right\}, \qquad b \in \mathcal{L}_t. \tag{16}$$

The corrected forward derivative satisfies $D_b^F V_{i,t}(b) \geq D_b^B V_{i,t}(b)$. Since consumption is decreasing in the marginal value of wealth,

$$c_{i,t}^F(b) \leq c_{i,t}^B(b), \qquad S_{i,t}^F(b) \geq S_{i,t}^B(b) = \widetilde{S}_{i,t}(b).$$

Hence, the forward policy selected for a positive drift satisfies the state boundary constraint, while the correction is inactive whenever the unconstrained policy implies faster repayment. This preserves consistency between the drift direction and the derivative selected by the upwind scheme.[5] The terminal stationary problem is solved on the original grid, retaining the constraints in (13) below $\underline{b}_T$ even though the terminal invariant distribution assigns no mass there. The resulting terminal value function initializes the backward HJB iteration, whose policy generators then evolve the initial density forward through (8).

4 Numerical illustration

We use the calibration in Obiols-Homs (2011), with a small adjustment to the discount rate to reproduce their equilibrium in continuous time.[6] The parameter values are $\sigma = 2$, $\rho = 0.12145$, $\alpha = 0.36$, $A = 0.273$, and $\delta = 0.08$. The labor efficiency levels are $z_L = 1, z_M = 5.29$ and $z_H = 46.55$. The continuous-time transition rates are calibrated to match those in Obiols-Homs (2011). To illustrate the method, we impose a constant minimum drift $\widetilde{S}_{i,t}(b) = 0.0001$ in $\mathcal{L}_t$ and report sensitivity to this implementation choice. The model is solved on a non-uniform wealth grid with 401 points and a step-size growth parameter of 0.01.[7]

Figure 1 summarizes the results. Panel (a) plots the wealth CDF at selected dates during a benchmark contraction from $\underline{b}_0 = -1$ to $\underline{b}_T = -0.5$. Panel (b) reports the associated deviations of the wage and interest rate from their initial steady-state values. Panel (c) shows steady-state consumption equivalent gains (CEG), relative to $\underline{b}_0$, across terminal borrowing limits.[8] Panel (d) reports transition welfare in general equilibrium and with prices fixed at their initial steady-state values for $\widetilde{S} \in \{0.0001, 0.0005\}$. The partial equilibrium losses isolate the direct cost of restricting household savings choices, whereas the difference between the general and partial equilibrium curves reflects the contribution of equilibrium price adjustment.

[5] Under concavity, the usual drift-sign conditions provide the familiar upwind characterization. Our implementation instead compares the Hamiltonians of the forward, backward, and zero-drift candidates. In $\mathcal{L}_t$, the derivative correction ensures that the admissible forward candidate satisfies the imposed drift floor.

[6] The underlying calibration is also used in Dávila et al. (2012).

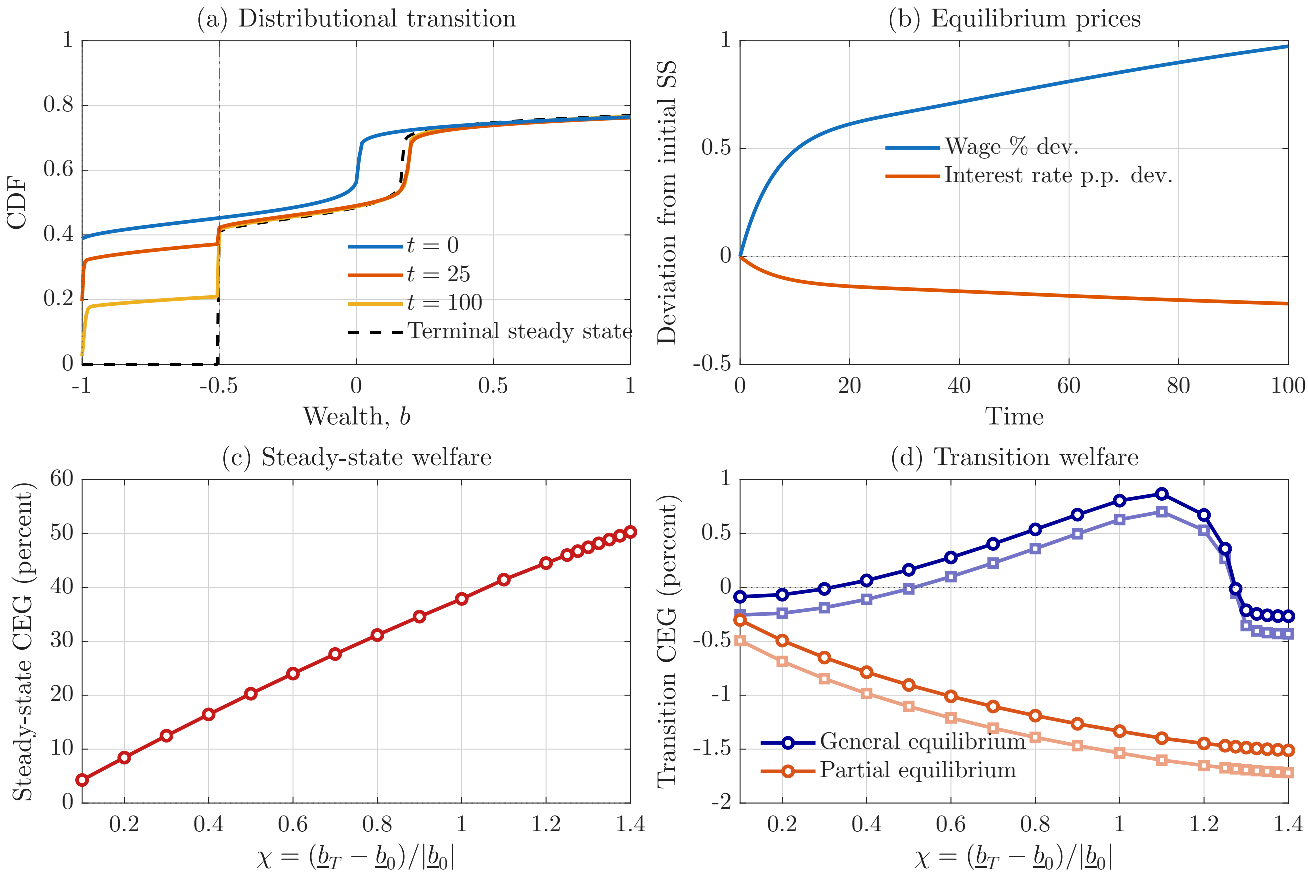
[7] As a validation exercise, the replication package shows that we obtain the same results with finer time and wealth grids.

[8] CEG is computed as

$$\text{CEG} = \left[\left(\frac{\mathcal{W}_c}{\mathcal{W}_0}\right)^{\frac{1}{1-\sigma}} - 1\right] * 100, \tag{17}$$

where $\mathcal{W}_0$ denotes aggregate expected lifetime utility in the initial steady state with debt limit $\underline{b}_0$, and $\mathcal{W}_c$ denotes aggregate expected lifetime utility under the tighter limit $\underline{b}_T$. For steady-state comparisons, $\mathcal{W}_c$ is evaluated in the terminal steady state; for transition comparisons, it is evaluated at t_0^+ using the initial distribution.

Fig. 1: Debt Limit Contractions: Distributional Dynamics, Prices, and Welfare



Note: In panel (d) dark (light) curves with circular (square) markers use $\widetilde{S} = 0.0001$ ($\widetilde{S} = 0.0005$).

A tighter debt limit raises aggregate capital, increasing the wage and lowering the interest rate. With prices held fixed, contractions are uniformly welfare reducing because they restrict household choices and require agents with previously admissible debt to deleverage. In general equilibrium, the resulting price changes benefit households with little or negative wealth, whose marginal utility is comparatively high.[9]

Let $\chi = (\underline{b}_T - \underline{b}_0)/|\underline{b}_0|$ denote the size of the contraction. Panel (c) shows that steady-state CEG rises with the size of the contraction. Transition welfare in panel (d), by contrast, is hump-shaped in general equilibrium. As χ initially rises, the constraint affects mainly low-income borrowers, while middle-income households respond by saving more rapidly to exit $\mathcal{L}_t$, strengthening capital accumulation and the associated price gains. Beyond a sufficiently large contraction, however, the state boundary condition applies to a much broader set of households, and many middle-income agents no longer choose the accelerated saving path, instead saving at the imposed minimum $\widetilde{S}$. The general equilibrium welfare effect is therefore hump-shaped. A higher minimum drift, $\widetilde{S}$, accelerates deleveraging but raises its direct consumption cost, increasing partial equilibrium losses and attenuating general equilibrium gains, as shown by the lighter curves in panel (d). Results therefore depend on the calibrated income process and on $\widetilde{S}$.

[9] The welfare losses are concentrated among households at or close to the initial debt limit, especially those in the lowest labor income state, because they must reduce consumption to generate the required positive drift. Low-wealth households outside $\mathcal{L}_t$ are less directly affected and benefit from higher wages. Wealthy agents, who derive relatively more of their income from assets, lose out due to the decline in the interest rate. Since the stationary distribution contains relatively little mass at very high wealth, their losses receive less weight in aggregate welfare.

5 Conclusion

This note develops a fixed-grid method for computing transitions after a contraction of the borrowing limit in continuous-time heterogeneous-agent models. Extended state boundary conditions apply to households with legacy debt, while a modification of the upwind scheme ensures that their policy functions satisfy the required wealth drift. The initial and terminal steady states, together with the transitional HJB–KFE system, can therefore be solved on a common grid. In the numerical application, general equilibrium changes in wages and interest rates can outweigh the direct cost of deleveraging and generate positive aggregate transition welfare for moderate contractions. The application thereby complements the welfare analysis of Obiols-Homs (2011) by accounting for the transition costs of debt limit contractions. These welfare effects nevertheless depend on the severity of the contraction, the calibrated income process, and the treatment of legacy debt during the transition.

Data availability

Replication code is available at `https://github.com/GMellior/debt-contractions-hact`.

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